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Research Article

Solution of Nonlinear Volterra Integral Equations with Weakly Singular Kernel by Using the HOBW Method

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We present a new numerical technique to discover a new solution of Singular Nonlinear Volterra Integral Equations (SNVIE). The considered technique utilizes the Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet method (HOBW) to solve the weakly SNVIE including Abel's equations. We acquire the HOBW implementation matrix of the integration to derive the procedure of solving these kind integral equations. The explained technique is delineated with two numerical cases to demonstrate the benefit of the technique used by us. At last, the exchange uncovers the way that the strategy utilized here is basic in usage.

1. Introduction

In the current literature, there are many different applications of SNVIE in various areas, such as mathematical physics, electrochemistry, scattering theory, heat conduction, semiconductors, population dynamics, and fluid flow [1, 2]. Numerical strategies for the SNVIE are spline collocation methods [3], Newton–Cotes methods [4], extrapolation algorithm [5], and Hermite-collocation method [6]. The most popular methods for talking about the such equations are introduced, such as homotopy asymptotic method [7], Nyström interpolant method [8], Mesh method [9], Tau method [10], Laplace transform [11], orthonormal Bernstein, and block-pulse functions [12–17].

Wavelet theory is a moderately new and considered as a rising territory in the field of applied science and engineering. Wavelets allow the accurate representation of a lot of functions. The wavelet technique is a new numerical technique utilized for dissolving the fractional equations. SNVIE has numerous applications in different zones, for example, semiconductors' mathematical chemistry, chemical reactions, physics, scattering theory, electrochemistry, seismology, metallurgy, fluid flow, and population dynamics [2, 18–20].

In 1823, Niels Henrik Abel derived the equation

$$f(x) + \int_0^x \frac{u_t}{(x-t)^{0.5}}\,dt = 0, \tag{1}$$

where $u(t)$ is an unknown function and $f(x)$ is a given function. This equation is an example of a nonhomogeneous Volterra equation of first kind with weak singularity. Abel obtained this equation while studying the motion of a particle on a smooth curve lying on a vertical plane. The physical depiction of this condition is given in [21] as pursues. Abel thought about the issue in traditional mechanics, which is that of deciding the time a molecule brings to slide openly down a smooth settled bend in a vertical xy-plane (in

Figure 1), from any settled point (X, Y) on the bend to its absolute bottom (the starting point 0). If m means the mass of the molecule and $xp(y)$ signifies the condition of the smooth bend where x is a differentiable function of y, at that point we acquire the vitality protection condition as

$$\frac{1}{2}mv^2 + mgy = mgY, \tag{2}$$

where v is the speed of the molecule at the position (x, y) at time t, if the molecule tumbles from rest at time $t = 0$ from the point (x, y), and g represents acceleration due to gravity. The connection (2) can be expressed as

$$\frac{ds}{dt} = -\left[2g(Y - y)\right]^{1/2} \tag{3}$$

by utilizing the arc-length $s(t)$, estimated from the starting point to the point(x, y), where a less sign has been utilized in the square root since s diminishes with time t amid the fall of the molecule. Using the formula

$$\frac{ds}{dy} = \left[1 + p'^2(y)\right]^{0.5} \tag{4}$$

we can compose

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = -\left[\frac{2g(Y - y)}{1 + p'^2(y)}\right]^{0.5} \tag{5}$$

By integrating both sides of (5), we obtain

$$\int_Y^0 \left[\frac{1 + p'^2(y)}{2g(Y - y)^{0.5}}\right] dy = -\int_0^T dy = -T, \tag{6}$$

where T is the total time of fall of the particle, from the point (x, y) to the origin $(0, 0)$. Therefore, we have

$$\int_0^Y \frac{u(y)}{(Y - y)^{0.5}} dy = T = f(Y),$$
$$u(y) = \left[\frac{1 + p'^2(y)}{2g}\right]^{0.5}, \tag{7}$$
$$0 < Y < a$$

where $f(0) = 0$. In this way, we can find that the time of descent of the particle, T, can be resolved totally by utilizing the recipe (7), if the state of the curve $x\ p(y)$, and consequently the function $u(y)$ is known. On the off chance that we consider, on the other hand, the issue of assurance of the state of the bend, when the time of fall T is known, which is the historic Abel's problem, then the relation (7) is an integral equation for the unknown function $u(y)$, which is known as Abel's integral equation.

The most general form of Abel's integral equation is given by

$$\int_0^x \frac{u(t)}{[h(x) - h(t)]^{\alpha}} dt = f(x),$$
$$x > 0,\ f(0) = 0,\ 0 < \alpha < 1, \tag{8}$$

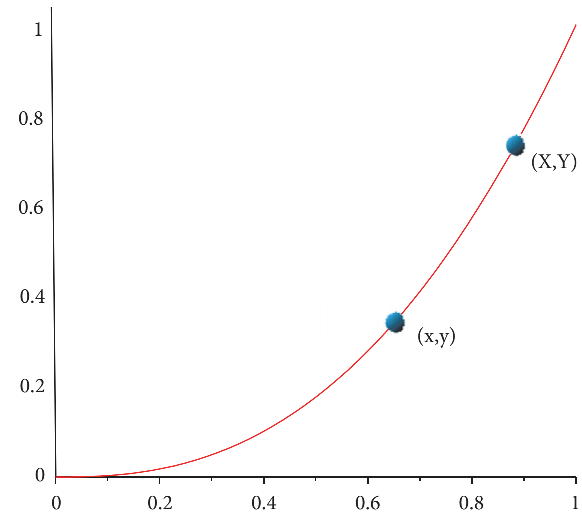

Figure 1: Abel's problem.

where $h(x)$ is a monotonically expanding function. We have picked it as $h(x)$. Also, a general form of SVIE of second kind is given as

$$u(x) = f(x) + \lambda \int_0^x \frac{u(t)}{(x - t)^{\alpha}} dt, \quad 0 < \alpha < 1, \tag{9}$$

where $f(x)$ is in $L^2(R)$ on the interim $t \le T,\ 0 \le x$ and λ is a steady parameter.

We utilize the HOBW method for determining the approximation solution of SNVIE of the shape given by

$$y(x) = f(x) + \lambda \int_a^x \frac{k(x, t) F(y(t))}{(x - t)^{1-\alpha}} dt, \quad 0 \le x \le 1. \tag{10}$$

where $f(x),\ k(x, t)$ are continuous functions, while $0 < \alpha < 1$ and $y(x)$ is the unknown function to be determined.

This paper is organized as follows. Initially the basic formulation of the HOBW method and some properties of HOBW are defined in Section 2. In Section 3, we determine the HOBW implementation matrix of integration. While in Section 4, we summarize the process of dissolving weakly singular-Volterra integral equations based on the HOBW implementation matrix method. In Section 5, we consider two examples which demonstrate the validity of this method. Finally, the concluding remarks are demonstrated.

2. The HOBW Method and Operational Matrix of the Integration

2.1. Wavelets and the HOBW Method. Wavelets constitute a group of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet. In which parameter of dilation a and parameter of translation b vary continuously.

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t - b}{a}\right), \quad a, b \in R,\ a \neq 0 \tag{11}$$

By letting a and b be discrete values such as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1, b_0 > 0$,

where n and k are positive integers, we attain the family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{k/2}\,\psi\left(a_0^{\,k} t - nb_0\right), \quad n, k \in Z^+ \tag{12}$$

Then we see that $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$, $b_0 = 1$, then $\psi_{k,n}(t)$ forms an orthonormal basis. Here, $HOBW_{i,j}(t) = HOBW(k, i, j, t)$ involves four arguments, $i = 1, \ldots, 2^{k-1}$, k is to be any positive integer, j is the degree of the Bernstein polynomials, and t is the normalized time. $HOBW_{i,j}(t)$ are defined on $[0, 1)$ as [12, 13]

$$HOBW_{i,j}(t) = \begin{cases} 2^{(k-1)/2}\binom{n}{j}\left(2^{k-1}x - i + 1\right)^j\left(1 - \left(2^{k-1}x - i + 1\right)\right)^{n-j} & \dfrac{i-1}{2^{k-1}} \le t < \dfrac{i}{2^{k-1}} \\ 0 & otherewise \end{cases} \tag{13}$$

where $i = 1, 2, \ldots, 2^{k-1}$, $j = 0, 1, \ldots, M-1$ and k is a positive integer. Thus, we attain our new basis as $\{\mathrm{HOBW}_{1,0}, \mathrm{HOBW}_{1,1}, \ldots, \mathrm{HOBW}_{2^{k-1},M-1}\}$ and any function is truncated with them.

The $HOBW$ detect orthonormal basis is given by

$$\left(HOBW_{ij}(x), HOBW_{i'j'}(x)\right) = \begin{cases} 1 & (i,j) = \left(i', j'\right) \\ 0 & (i,j) \neq \left(i', j'\right) \end{cases} \tag{14}$$

where $(\cdot,\cdot)$ is called the inner product in $L^2[0, 1)$. The HOBW has compact support $[(i-1)/2^{k-1}, i/2^{k-1}]$, $i = 1, \ldots, 2^{k-1}$.

2.2. Function Approximation by Using the HOBW Functions. Any function $y(t)$, which is integrable in $[0, 1)$, is truncated by using the HOBW method as follows:

$$\begin{gathered} y(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij} HOBW_{ij}(t), \\ i = 1, 2, \ldots, \infty, \quad j = 0, 1, 2, \ldots, \infty, \quad t \in [0, 1), \end{gathered} \tag{15}$$

where the HOBW coefficients c_{ij} can be calculated as given below:

$$c_{ij} = \frac{\left(y(t), \mathrm{HOBW}_{ij}(t)\right)}{\left(\mathrm{HOBW}_{ij}(t), \mathrm{HOBW}_{ij}(t)\right)} \tag{16}$$

We approximate $y(t)$ by a truncated series as follows:

$$y(t) = \sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M-1} c_{ij} HOBW_{ij}(t) = C^T \mathrm{HOBW}(t) \tag{17}$$

where $\mathrm{HOBW}(t)$ and C are $2^{k-1}M \times 1$ vectors given by

$$\begin{aligned} \mathrm{HOBW}(t) = \big[&\mathrm{HOBW}_{10}, \mathrm{HOBW}_{11}, \ldots, \mathrm{HOBW}_{1(M-1)}, \\ &\mathrm{HOBW}_{20}, \mathrm{HOBW}_{21}, \ldots, \mathrm{HOBW}_{2(M-1)}, \ldots, \\ &\mathrm{HOBW}_{2^{k-1}0}, \ldots, \mathrm{HOBW}_{2^{k-1}(M-1)}\big]^T. \end{aligned} \tag{18}$$

and

$$\begin{aligned} C = \big[&c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, c_{21}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, \\ &c_{2^{k-1}(M-1)}\big]^T. \end{aligned} \tag{19}$$

We define the HOBW matrix $\Phi_{2^{k-1}M \times 2^{k-1}M}$ as follows:

$$\begin{aligned} \Phi_{2^{k-1}M \times 2^{k-1}M} = \Bigg[&\mathrm{HOBW}\left(\frac{1}{2\cdot 2^{k-1}M}\right), \\ &\mathrm{HOBW}\left(\frac{3}{2\cdot 2^{k-1}M}\right), \ldots, \\ &\mathrm{HOBW}\left(\frac{\left(\left(2\cdot 2^{k-1}M\right) - 1\right)}{2\cdot 2^{k-1}M}\right)\Bigg], \end{aligned} \tag{20}$$

The series in (17) contains an infinite number of terms for a smooth function $y(t)$. Therefore, we have

$$C^T \langle \mathrm{HOBW}(t), \mathrm{HOBW}(t)\rangle = \langle y(t), \mathrm{HOBW}(t)\rangle \tag{21}$$

so that

$$C = D^{-1}\langle y(t), \mathrm{HOBW}(t)\rangle, \tag{22}$$

where

$$D = \langle \mathrm{HOBW}(t), \mathrm{HOBW}(t)\rangle, \tag{23}$$

$$= \int_0^1 \mathrm{HOBW}(t) . \mathrm{HOBW}^T(t)\, dt \tag{24}$$

$$= \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & D_M \end{pmatrix} \tag{25}$$

Then, by using (14), D_i ($i = 1, 2, \ldots, 2^{k-1}$) is defined as follows:

$$\begin{aligned} \left(D_n\right)_{i+1,j+1} = &\int_{(i-1)/2^{k-1}}^{i/2^{k-1}} \mathrm{HOBW}_{i,n}\left(2^{k-1}t - i + 1\right) \\ &\cdot \mathrm{HOBW}_{j,n}\left(2^{k-1}t - i + 1\right) dt \end{aligned} \tag{26}$$

$$\left(D_n\right)_{i+1,j+1} = \frac{1}{2^{k-1}}\int_0^1 \mathrm{HOBW}_{i,n}(t)\,\mathrm{HOBW}_{j,n}(t)\, dt \tag{27}$$

$$\left(D_n\right)_{i+1,j+1} = \frac{\binom{n}{i}\binom{n}{j}}{2^{k-1}(2n+1)\binom{2n}{i+j}} \tag{28}$$

We can also approximate the function $k(x,t) \in L[0,1]$ as follows:

$$k(x,t) \approx \mathrm{HOBW}^T(x)\, K\, \mathrm{HOBW}(t), \tag{29}$$

where K is an $2^{k-1}M \times 2^{k-1}M$ matrix that we attain as follows:

$$K = D^{-1} \langle \mathrm{HOBW}(x) \langle k(x,t), \mathrm{HOBW}(t) \rangle \rangle D^{-1} \tag{30}$$

2.3. Multiplication of the Hybrid Functions. We can evaluate $\mathrm{HOBW}_{(2^{k-1}M\times 1)}(t)\mathrm{HOBW}^T{}_{(2^{k-1}M\times 1)}(t)$ for VIE of the second kind via the HOBW functions as detailed below.

Let the product of $\mathrm{HOBW}_{(2^{k-1}M\times 1)}(t)$ and $\mathrm{HOBW}^T{}_{(2^{k-1}M\times 1)}(t)$ be given by

$$\begin{aligned} &\mathrm{HOBW}_{(2^{k-1}M\times 1)}(t)\, \mathrm{HOBW}^T{}_{(2^{k-1}M\times 1)}(t) \\ &\quad \cong M_{(2^{k-1}M\times 2^{k-1}M)}(t) \end{aligned} \tag{31}$$

where

$$\begin{aligned} &M_{(2^{k-1}M\times 2^{k-1}M)}(t) \\ &= \begin{bmatrix} \mathrm{HOBW}_{10}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{10}(t)\,\mathrm{HOBW}_{20}(t) & \cdots & \mathrm{HOBW}_{10}(t)\,\mathrm{HOBW}_{2^{k-1},M}(t) \\ \mathrm{HOBW}_{20}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{20}(t)\,\mathrm{HOBW}_{20}(t) & \cdots & \mathrm{HOBW}_{20}(t)\,\mathrm{HOBW}_{2^{k-1},M}(t) \\ \mathrm{HOBW}_{30}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{30}(t)\,\mathrm{HOBW}_{20}(t) & \cdots & \mathrm{HOBW}_{30}(t)\,\mathrm{HOBW}_{2^{k-1},M}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{HOBW}_{2^{k-1},M}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{2^{k-1},M}(t)\,\mathrm{HOBW}_{20}(t) & \cdots & \mathrm{HOBW}_{2^{k-1},M}(t)\,\mathrm{HOBW}_{2^{k-1},M}(t) \end{bmatrix} \end{aligned} \tag{32}$$

With the recursive formulas, we calculate $M_{(2^{k-1}M\times 2^{k-1}M)}(t)$ for any k and M.

The matrix $M_{(2^{k-1}M\times 2^{k-1}M)}(t)$ in (23) satisfies the following relation:

$$\begin{aligned} &M_{(2^{k-1}M\times 2^{k-1}M)}(t)\, c_{(2^{k-1}M\times 1)} \\ &= \widetilde{C}_{(2^{k-1}M\times 2^{k-1}M)} \mathrm{HOBW}_{(2^{k-1}M\times 1)}(t) \end{aligned} \tag{33}$$

where $c_{(2^{k-1}M)}$ is defined in (33) and $\widetilde{C}_{(2^{k-1}M\times 2^{k-1}M)}$ is the matrix coefficient. We consider the case when $k = 3$ and $M = 4$. Thus, we have

$$M_{(16)\times 16)}(t) = \begin{bmatrix} \mathrm{HOBW}_{10}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{10}(t)\,\mathrm{HOBH}_{20}(t) & \cdots & \mathrm{HOBW}_{10}(t)\,\mathrm{HOBW}_{43}(t) \\ \mathrm{HOBW}_{20}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{20}(t)\,\mathrm{HOBH}_{20}(t) & \cdots & \mathrm{HOBW}_{20}(t)\,\mathrm{HOBW}_{43}(t) \\ \mathrm{HOBW}_{30}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{30}(t)\,\mathrm{HOBH}_{20}(t) & \cdots & \mathrm{HOBW}_{30}(t)\,\mathrm{HOBW}_{43}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{HOBW}_{43}(t)\,\mathrm{HOBW}_{10}(t) & \mathrm{HOBW}_{43}(t)\,OBH_{20}(t) & \cdots & \mathrm{HOBW}_{43}(t)\,\mathrm{HOBW}_{43}(t) \end{bmatrix} \tag{34}$$

The coefficient matrix $\widetilde{C}_{(2^{k-1}M\times 2^{k-1}M)}$ in (33) is determined by

$$\widetilde{C}_{(2^{k-1}M\times 2^{k-1}M)} = \begin{bmatrix} C_0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix} \tag{35}$$

where $C_i,\ i = 0, 1, 2, 3$ are 4×4 matrices given by

$$C_{i((2^{k-1}M\times 2^{k-1}M))}=\begin{bmatrix} \frac{7}{3}\mathbf{c}_{1i}+\frac{5}{23}\mathbf{c}_{2i} & \frac{11}{16}\mathbf{c}_{1i}+\frac{3}{11}\mathbf{c}_{2i} & \frac{3}{22}\mathbf{c}_{1i}+\frac{5}{16}\mathbf{c}_{2i} & \frac{-3}{19}\mathbf{c}_{1i}+\frac{1}{19}\mathbf{c}_{2i} \\ -\frac{1}{103}\mathbf{c}_{3i}-\frac{7}{201}\mathbf{c}_{4i} & +\frac{5}{21}\mathbf{c}_{3i}+\frac{1}{101}\mathbf{c}_{4i} & -\frac{1}{44}\mathbf{c}_{3i}+\frac{1}{103}\mathbf{c}_{4i} & +\frac{3}{209}\mathbf{c}_{3i}-\frac{3}{209}\mathbf{c}_{4i} \\ \frac{22}{144}\mathbf{c}_{1i}+\frac{1}{23}\mathbf{c}_{2i} & \frac{22}{23}\mathbf{c}_{1i}+\frac{14}{123}\mathbf{c}_{2i} & \frac{4}{19}\mathbf{c}_{1i}+\frac{2}{13}\mathbf{c}_{2i} & \frac{-1}{23}\mathbf{c}_{1i}+\frac{1}{19}\mathbf{c}_{2i} \\ -\frac{17}{143}\mathbf{c}_{3i}-\frac{13}{209}\mathbf{c}_{4i} & +\frac{15}{47}\mathbf{c}_{3i}+\frac{2}{175}\mathbf{c}_{4i} & -\frac{2}{81}\mathbf{c}_{3i}+\frac{2}{147}\mathbf{c}_{4i} & +\frac{1}{190}\mathbf{c}_{3i}-\frac{3}{105}\mathbf{c}_{4i} \\ \frac{9}{14}\mathbf{c}_{1i}+\frac{7}{19}\mathbf{c}_{2i} & \frac{24}{25}\mathbf{c}_{1i}+\frac{7}{11}\mathbf{c}_{2i} & \frac{4}{15}\mathbf{c}_{1i}+\frac{24}{132}\mathbf{c}_{2i} & \frac{-19}{142}\mathbf{c}_{1i}+\frac{3}{19}\mathbf{c}_{2i} \\ -\frac{13}{132}\mathbf{c}_{3i}-\frac{11}{120}\mathbf{c}_{4i} & +\frac{7}{34}\mathbf{c}_{3i}+\frac{3}{107}\mathbf{c}_{4i} & -\frac{5}{71}\mathbf{c}_{3i}+\frac{3}{107}\mathbf{c}_{4i} & +\frac{3}{201}\mathbf{c}_{3i}-\frac{3}{208}\mathbf{c}_{4i} \\ \frac{11}{13}\mathbf{c}_{1i}+\frac{1}{20}\mathbf{c}_{2i} & \frac{22}{23}\mathbf{c}_{1i}+\frac{13}{27}\mathbf{c}_{2i} & \frac{7}{23}\mathbf{c}_{1i}+\frac{2}{13}\mathbf{c}_{2i} & \frac{-3}{23}\mathbf{c}_{1i}+\frac{3}{25}\mathbf{c}_{2i} \\ -\frac{3}{103}\mathbf{c}_{3i}-\frac{1}{209}\mathbf{c}_{4i} & +\frac{7}{37}\mathbf{c}_{3i}+\frac{2}{103}\mathbf{c}_{4i} & -\frac{1}{90}\mathbf{c}_{3i}+\frac{3}{103}\mathbf{c}_{4i} & +\frac{1}{213}\mathbf{c}_{3i}-\frac{1}{201}\mathbf{c}_{4i} \end{bmatrix} \quad (36)$$

3. HOBW Operational Matrix

Firstly, we review some basic definitions of fractional calculus [22–24], which are required for establishing our results.

Definition 1. The Riemann–Liouville fractional integral operator I of order α, of a function $f \in C_\nu$, $\nu \geq -1$, is defined as follows:

$$(I^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, & \alpha>0,\ t>0, \\ f(t), & \alpha=0 \end{cases} \quad (37)$$

The block-pulse functions (BPFs), an m-set of BPFs on $[0,1)$, are defined by

$$b_i(t) = \begin{cases} 1, & \frac{i}{m}<t<\frac{i+1}{m}, \\ 0, & otherwise, \end{cases} \quad (38)$$

where $i = 0,1,2,\ldots,m-1$. The BPFs have the orthogonal properties as follows:

$$b_i(t)\,b_j(t) = \begin{cases} 0, & i\neq j, \\ b_i(t), & i=j, \end{cases} \quad (39)$$

and

$$\int_0^1 b_i(\tau)\,b_j(\tau) = \begin{cases} 0, & i\neq j, \\ 1, & i=j, \end{cases} \quad (40)$$

Every function $f(t)$ which is integrable in $[0,1)$ can be truncated with the aid of BPFs series as

$$f(t) \approx \sum_{i=0}^{m-1} f_i b_i(t) = F^T B_m(t), \quad (41)$$

where $F = [f_0, f_1, \ldots, f_{m-1}]^T$, $B_m(t) = [b_0, b_1, \ldots, b_{m-1}]^T$.

Using the disjointness of BPFs and the matrix of $B_m(t)$ can be gotten by

$$B_m(t)\,B_m^T(t) = \begin{bmatrix} b_0(t) & & 0 \\ & b_1(t) & \\ & & \ddots & \\ 0 & & & b_{m-1}(t) \end{bmatrix} \quad (42)$$

Equation (41) implies that the HOBW method can be truncated into an m-set BPFs as follows:

$$HOBW_{2^{k-1}M} = \Phi_{2^{k-1}M\times 2^{k-1}M} B_{2^{k-1}M}(t). \quad (43)$$

The block-pulse implementation matrix of the fractional integration F^α has been given in [14] as follows:

$$(I^\alpha B_{2^{k-1}M})(t) \approx F^\alpha B_{2^{k-1}M}(t) \quad (44)$$

where

$$F^\alpha = \frac{1}{(2^{k-1}M)^\alpha}\frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{m'-1} \\ 0 & 1 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m'-2} \\ 0 & 0 & 1 & \zeta_1 & \cdots & \zeta_{m'-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \zeta_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad (45)$$

$$\zeta_k = (k+1)^{\alpha+1} - (2k)^{\alpha+1} + (k-1)^{\alpha+1}. \quad (46)$$

At $\alpha = 1$, F^α is BPF's implementation matrix of integration.

Let

$$(I^\alpha \psi_{2^{k-1}M})(t) \approx \mathrm{P}^\alpha_{2^{k-1}M\times 2^{k-1}M} B_{2^{k-1}M}(t) \quad (47)$$

where the matrix $\mathrm{P}^{\alpha}_{2^{k-1}M\times 2^{k-1}M}$ is called the HOBW implementation matrix of fractional integration [2, 17]. Using (43) and (44), we have

$$\begin{aligned}\left(I^{\alpha}\psi_{2^{k-1}M}\right)(t) &\approx \left(I^{\alpha}\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}\right)(t)\\ &= \Phi_{2^{k-1}M\times 2^{k-1}M}\left(I^{\alpha}B_{2^{k-1}M}\right)(t)\\ &= \Phi_{2^{k-1}M\times 2^{k-1}M}F^{\alpha}B_{2^{k-1}M}(t).\end{aligned} \quad (48)$$

From (38) and (39) we can get

$$\begin{aligned}&\mathrm{P}^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\psi_{2^{k-1}M}(t)\\ &\quad= \mathrm{P}^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(t)\\ &\quad= \Phi_{2^{k-1}M\times 2^{k-1}M}F^{\alpha}B_{2^{k-1}M}(t)\end{aligned} \quad (49)$$

Then the matrix is $\mathrm{P}^{\alpha}_{2^{k-1}M\times 2^{k-1}M}$ given by

$$P^{\alpha}_{2^{k-1}M\times 2^{k-1}M} = \Phi_{2^{k-1}M\times 2^{k-1}M}F^{\alpha}\Phi^{-1}_{2^{k-1}M\times 2^{k-1}M} \quad (50)$$

For example, when $\alpha = 0.5$, $M = 2$, and $k = 3$, the operational matrix of the fractional integration $\mathrm{P}^{\alpha}_{2^{k-1}M\times 2^{k-1}M}$ is expressed as follows:

$$\mathrm{P}^{0.5}_{8\times 8} = \begin{bmatrix} 0.19343 & 0.2579 & 0.1725 & 0.10262 & 0.10856 & 0.08539 & 0.08608 & 0.07330\\ -0.0645 & 0.5158 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.19344 & 0.25792 & 0.17247 & 0.10262 & 0.10856 & 0.08539\\ 0 & 0 & -0.04299 & 0.34389 & 0.24862 & 0.09946 & 0.12097 & 0.09077\\ 0 & 0 & 0 & 0 & 0.19344 & 0.25792 & 0.17247 & 0.10262\\ 0 & 0 & 0 & 0 & -0.04299 & 0.34389 & 0.24864 & 0.09946\\ 0 & 0 & 0 & 0 & 0 & 0 & 0.19344 & 0.25792\\ 0 & 0 & 0 & 0 & 0 & 0 & -0.04299 & 0.34389 \end{bmatrix} \quad (51)$$

4. Solution of Nonlinear Volterra Integral Equations via the HOBW Method

Consider the following integral equation:

$$y(x) = f(x) + \lambda\int_0^x \left(\frac{k(x,t)\left(y(t)\right)^q}{(x-t)^{1-\alpha}}dt, \quad 0 \le x \le 1, \quad (52)$$

$$\begin{aligned}&y(x) \approx Y^T HOBW(x) \approx Y^T\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(x)\\ &k(x,t) \approx HOBW^T(x)\,K\,HOBW(t)\\ &\quad\approx \left(\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(x)\right)^T\\ &\quad\cdot K\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(t)\\ &f(x) \approx F^T HOBW(x) \approx F^T\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(x)\end{aligned} \quad (53)$$

where K is $2^{k-1}M \times 2^{k-1}M$ matrix with

$$K = \left(HOBW(x), \left(k(x,t), HOBW(t)\right)\right). \quad (54)$$

The functions $y^q(x)$ can be truncated into the HOBB functions as

$$\begin{aligned}y^2(t) &= \left[Y^T\mathrm{HOBW}(t)\right]^2\\ &= Y^T\mathrm{HOBW}(t)\,\mathrm{HOBW}(t)^T Y\\ &= \mathrm{HOBW}(t)^T\,\widetilde{Y}Y\\ y^3(t) &= Y^T\mathrm{HOBW}(t)\left[Y^T\mathrm{HOBW}(t)\right]^2\\ &= Y^T\mathrm{HOBW}(t)\,\mathrm{HOBW}(t)^T\,\widetilde{Y}Y\\ &= \mathrm{HOBW}(t)^T\,\widetilde{Y}\widetilde{Y}Y\\ &= \left(\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(t)\right)^T\left(\widetilde{Y}\right)^2 Y\\ y^q(t) &= \mathrm{HOBW}(t)^T\left(\widetilde{Y}\right)^{q-1}Y\\ &= \left(\Phi_{2^{k-1}M\times 2^{k-1}M}B_{2^{k-1}M}(t)\right)^T\left(\widetilde{Y}\right)^{q-1}Y\end{aligned} \quad (55)$$

Therefore, upon substituting into (52), we get

$$\begin{aligned}Y^T HOBW(x) &= F^T HOBW(x) + \lambda\int_0^x (x-t)^{\alpha-1}\\ &\quad\cdot HOBW^T(x)\,K\,HOBW(t)\,\mathrm{HOBW}(t)^T\left(\widetilde{Y}\right)^{q-1}\\ &\quad\cdot Y\,dt\end{aligned} \quad (56)$$

$$\begin{aligned}Y^T HOBW(x) &= F^T HOBW(x) + \lambda HOBW^T(x)\\ &\quad\cdot K\int_0^x (x-t)^{\alpha-1} HOBW(t)\,\mathrm{HOBW}(t)^T\left(\widetilde{Y}\right)^{q-1}Y\,dt\end{aligned} \quad (57)$$

where

$$\begin{aligned}&\int_0^x (x-t)^{\alpha-1}\,\mathrm{HOBW}(t)\,\mathrm{HOBW}^T(t)\left(\widetilde{Y}\right)^{q-1}Y\,dt\\ &\quad= \int_0^x (x-t)^{\alpha-1}\left(\left(\widetilde{Y}\right)^{\widetilde{q}-1}Y\right)\mathrm{HOBW}(t)\,dt\end{aligned}$$

Table 1: Maximum absolute errors at different values of k and M for Example 1 via HOBW.

K	M=8	M=16	M=32
4	4.21×10^{-6}	4.10×10^{-10}	6.06×10^{-11}
5	3.71×10^{-7}	1.37×10^{-12}	2.92×10^{-12}
6	4.82×10^{-8}	3.27×10^{-13}	3.51×10^{-14}

Table 2: The comparison among HOBW, exact, and Chebyshev solutions for Example 2.

x_i	HOBW $k = 8,\ M = 8$	Exact Solution	Absolute error Of HOBW $k = 8,\ M = 8$	Absolute error Of Chebyshev base at $n = 8$
0.1	0.6309573449	0.6309573445	4×10^{-10}	1.4×10^{-5}
0.2	0.7247796642	0.7247796637	5×10^{-10}	2.1×10^{-5}
0.3	0.7860030896	0.7860030856	4×10^{-9}	3.5×10^{-5}
0.4	0.8325532052	0.8325532074	2.2×10^{-9}	5.5×10^{-5}
0.5	0.8705505674	0.8705505633	4.1×10^{-9}	2.4×10^{-5}
0.6	0.9028804552	0.9028804514	3.8×10^{-9}	4.4×10^{-5}
0.7	0.9311499137	0.9311499151	1.4×10^{-9}	5.2×10^{-5}
0.8	0.9563524927	0.9563524998	7.1×10^{-9}	7.1×10^{-5}
0.9	0.9791483643	0.9791483624	1.9×10^{-9}	6.9×10^{-5}

$$\begin{aligned} &= \Gamma(\alpha)\left(\left(\widetilde{Y}\right)^{\widetilde{q}-1} Y\right) I^{\alpha} B_{2^{k-1}M}(x) \\ &= \Gamma(\alpha)\left(\left(\widetilde{Y}\right)^{\widetilde{q}-1} Y\right) F^{\alpha} B_{2^{k-1}M}(x) \end{aligned} \tag{58}$$

With the aid of the previous equations, (52) becomes

$$\begin{aligned} &Y^T \Phi_{2^{k-1}M\times 2^{k-1}M} B_{2^{k-1}M}(x) \\ &\quad = F^T \Phi_{2^{k-1}M\times 2^{k-1}M} B_{2^{k-1}M}(x) \\ &\quad + \lambda\Gamma(\alpha)\left(\left(\widetilde{Y}\right)^{\widetilde{q}-1} Y\right) F^{\alpha} \Phi_{2^{k-1}M\times 2^{k-1}M} B_{2^{k-1}M}(x) \end{aligned} \tag{59}$$

where $HOBW(x) \approx \Phi_{2^{k-1}M\times 2^{k-1}M} B_{2^{k-1}M}(x)$.

To compute the unknown HOBW coefficients, we use the collocation points as follows:

$$t_i = \frac{2i-1}{2^{k-1}M}, \quad i = 1, 2, \ldots, 2^{k-1}M \tag{60}$$

From (60), we have a system of $2^{k-1}M$ nonlinear equations with $2^{k-1}M$ unknowns. Newton iteration method is used for completing the solution of the resulting nonlinear system, to get the unknown vectors Y. So, the approximated results $y(x)$ can be calculated as

$$y(x) = Y_{2^{k-1}M\times 1}{}^{T} HOBW_{2^{k-1}M\times 1}(x). \tag{61}$$

5. Numerical Examples

We use the demonstrated technique in this article for finding the numerical results of four weakly singular-Volterra integral equations.

Example 1. Consider the generalized Abel's integral equation [21].

$$u(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{u(t)}{\sqrt{(x-t)}}\,dt, \tag{62}$$

The exact solution is $u(x) = x^2$.

The outcomes demonstrate the high exactness and the effectiveness of the technique. This outcome can be effortlessly confirmed that the strategy yields the desired accuracy only in a few values of k and M. The results of this example at different values of k and M are presented in Table 1.

Example 2. Consider the following WSVIE:

$$u(x) = \frac{\pi x}{5}\csc\left(\frac{\pi}{5}\right) + x^{1/5} - \int_0^x \frac{u(t)}{(x-t)^{0.2}}\,dt, \tag{63}$$

The exact solution is $u(x) = \sqrt[5]{x}$.

Table 2 likewise checks all favorable circumstances of the strategy examined in the past examinations. It ought to be noticed that the HOBW additionally effortlessly composes PC code. This is another vital trademark for the numerical calculation. These actualities delineate the HOBW strategy as a quick, dependable, legitimate, and useful asset for understanding WSVIEs.

Example 3. Consider the singular kernel Volterra integral equation [25]:

$$\begin{aligned} y(x) &= f(x) + \int_0^x \frac{xt}{(x-t)^{0.5}} y^2(t)\,dt \\ f(x) &= x^3 - \frac{4096x^{17/2}}{6435} \end{aligned} \tag{64}$$

Table 3: The comparison among HOBW, exact, and SCW solutions for Example 3.

x_i	HOBW $k = 4,\ M = 2$	Exact Solution	Absolute error Of HOBW $k = 4,\ M = 2$	Absolute error Of SCW at $k = 4,\ M = 2$
0.1	0.0010010729	0.001	1.0729×10^{-6}	9.6679×10^{-5}
0.2	0.0080011451	0.008	1.145146×10^{-6}	4.7192×10^{-4}
0.3	0.0269910936	0.027	8.9064×10^{-6}	7.6218×10^{-4}
0.4	0.0639790239	0.064	2.0976×10^{-5}	5.2959×10^{-4}
0.5	0.1246774895	0.125	3.2251×10^{-4}	4.7042×10^{-3}
0.6	0.2157735888	0.216	2.2641×10^{-4}	4.3467×10^{-4}
0.7	0.3416681741	0.343	1.3318×10^{-3}	2.7024×10^{-5}
0.8	0.5074658635	0.512	4.5341×10^{-3}	5.0051×10^{-5}
0.9	0.7139664754	0.729	1.5033×10^{-3}	7.3236×10^{-3}

Table 4: The comparison among HOBW, analytic, and SCW solutions for Example 4.

x_i	Absolute error of HOBW $k = 4,\ M = 2$	Absolute error of SCW at $k = 4,\ M = 2$	Absolute error of HOBW $k = 5,\ M = 2$	Absolute error of SCW at $k = 5, M = 2$
0.1	4.1161×10^{-4}	6.8039×10^{-3}	2.413×10^{-4}	1.4565×10^{-3}
0.2	7.0253×10^{-4}	1.4873×10^{-3}	3.6508×10^{-5}	1.8367×10^{-4}
0.3	2.1435×10^{-4}	5.2635×10^{-4}	1.1701×10^{-5}	1.2211×10^{-4}
0.4	5.1621×10^{-5}	2.7043×10^{-4}	2.1061×10^{-4}	6.8020×10^{-5}
0.5	9.7003×10^{-5}	8.2247×10^{-4}	7.2708×10^{-5}	2.2140×10^{-4}
0.6	6.4325×10^{-4}	1.6089×10^{-4}	1.1578×10^{-6}	2.6100×10^{-5}
0.7	1.3045×10^{-4}	7.3143×10^{-5}	1.0186×10^{-6}	2.9370×10^{-5}
0.8	3.1324×10^{-3}	4.6887×10^{-5}	9.4106×10^{-4}	2.4144×10^{-6}
0.9	4.0371×10^{-3}	7.3699×10^{-5}	1.0512×10^{-4}	8.3715×10^{-6}

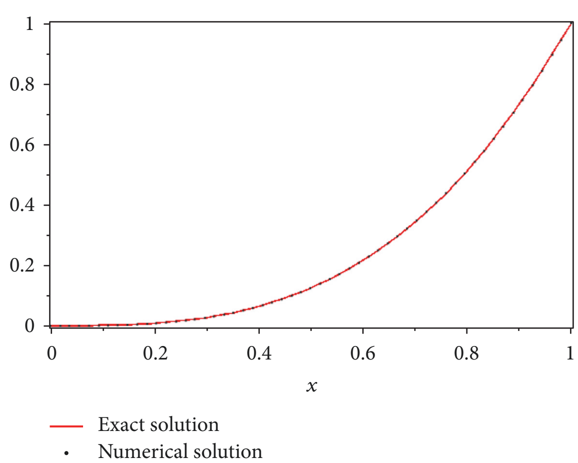

Figure 2: Comparison of numerical solutions and exact solution of Example 1 for $k = 4$, $M = 2$.

The analytic solution of (49) can be detected in [18] as $y(x) = x^3$.

The comparison among the *HOBW* solution and the second Chebyshev wavelet (SCW) solution is shown in Table 3 for $k = 4$ and $M = 2$, which confirms that the *HOBW* method gives almost the closer loose as the analytic solution. Figure 2 shows the comparison among the HOBW solution and the analytic one for $t \in [0, 1)$. Better approximation is expected by the values of k and M as in Table 2.

Example 4. Consider the nonlinear Volterra integral equation with singular kernel [25]:

$$\begin{aligned} y(x) &= f(x) + \int_0^x \frac{y^4(t)}{(x-t)^{0.5}}\,dt \\ f(x) &= \frac{\sqrt{x}}{15}\left(15 - 16x^2\right) \end{aligned} \tag{65}$$

with the exact solution $y(x) = \sqrt{x}$.

The comparison among the HOBW solution and the analytic solution for $t \in [0, 1)$ is shown in Table 4 and Figure 3 for $k = 4$ and $M = 2$ and confirms that the HOBW method gives almost the same solution as the analytic method. Better approximation is expected by choosing higher values of k and M.

6. Conclusion

In this investigation, the combination of orthonormal Bernstein, block-pulse functions, and wavelets is applied for resolving SNVIE. The main purpose of our method is to combine the orthonormal Bernstein and block-pulse functions wavelet method with the definition of the Riemann–Liouville fractional integral with the singular integral. The method

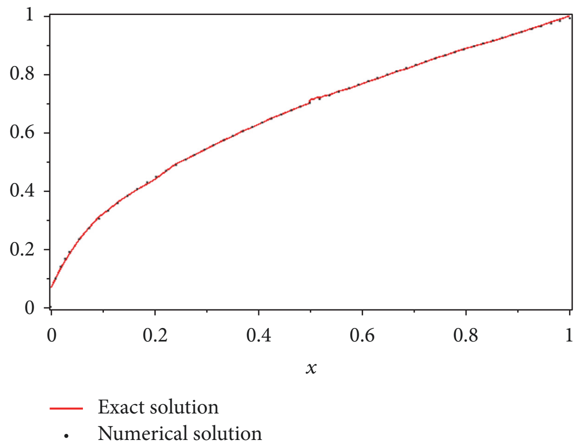

Figure 3: Results of Example 4 for $k = 4$, $M = 2$.

depends on reducing the considered system to a set of nonlinear algebraic equations. The generated system just needs sampling of functions and no integration. Wavelets as orthogonal systems have different resolution capability for truncating functions by the increasing of dilation parameter k that can give a good truncation for integral equations without using a polynomial solution. The considered method has its efficiency and simplicity. The matrices D and P are sparse; hence the CPU time and the computer memory will be reduced and at the same time the solution remains accurate. We also noted that when the degree of HOBW is increased, the errors will be decreased to smaller values. When the values of k and M are higher, we get more accurate solutions for the given problems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] P. K. Lamm and L. Eldn, "Numerical solution of first-kind Volterra equations by sequential Tikhonov regularization," *SIAM Journal on Numerical Analysis*, vol. 34, no. 4, pp. 1432–1450, 1997.

[2] H. J. Teriele, "Collocation methods for weakly singular second-kind Volterra integral equations with nonsmooth solution," *IMA (Institute of Mathematics and its Applications). Journal of Numerical Analysis*, vol. 2, no. 4, pp. 437–449, 1982.

[3] T. Diogo, "Collocation and iterated collocation methods for a class of weakly singular Volterra integral equations," *Journal of Computational and Applied Mathematics*, vol. 229, no. 2, pp. 363–372, 2009.

[4] T. Diogo, N. B. Franco, and P. Lima, "High order product integration methods for a Volterra integral equation with logarithmic singular kernel," *Communications on Pure and Applied Analysis*, vol. 3, no. 2, pp. 217–235, 2004.

[5] P. Lima and T. Diogo, "An extrapolation method for a Volterra integral equation with weakly singular kernel," *Applied Numerical Mathematics*, vol. 24, no. 2-3, pp. 131–148, 1997.

[6] T. Diogo and P. Lima, "Superconvergence of collocation methods for a class of weakly singular Volterra integral equations," *Journal of Computational and Applied Mathematics*, vol. 218, no. 2, pp. 307–316, 2008.

[7] M. S. Hashmi, N. Khan, and S. Iqbal, "Numerical solutions of weakly singular Volterra integral equations using the optimal homotopy asymptotic method," *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1567–1574, 2012.

[8] P. Baratella, "A Nyström interpolant for some weakly singular linear Volterra integral equations," *Journal of Computational and Applied Mathematics*, vol. 231, no. 2, pp. 725–734, 2009.

[9] J. Ma and Y. Jiang, "On a graded mesh method for a class of weakly singular Volterra integral equations," *Journal of Computational and Applied Mathematics*, vol. 231, no. 2, pp. 807–814, 2009.

[10] S. K. Vanani and F. Soleymani, "Tau approximate solution of weakly singular Volterra integral equations," *Mathematical and Computer Modelling*, vol. 57, no. 3-4, pp. 494–502, 2013.

[11] C. Yang, "An efficient numerical method for solving Abel integral equation," *Applied Mathematics and Computation*, vol. 227, pp. 656–661, 2014.

[12] M. A. Ramadan and M. R. Ali, "Numerical solution of volterra-fredholm integral equations using hybrid orthonormal bernstein and block-pulse functions," *Asian Research Journal of Mathematics*, vol. 4, no. 4, pp. 1–14, 2017.

[13] M. A. Ramadan and M. R. Ali, "Application of Bernoulli wavelet method for numerical solution of fuzzy linear Volterra-Fredholm integral equations," *Communication in Mathematical Modeling and Applications*, vol. 2, no. 3, pp. 40–49, 2017.

[14] M. A. Ramadan and M. R. Ali, "Solution of integral and Integro-Differential equations system using Hybrid orthonormal Bernstein and block-pulse functions," *Journal of abstract and computational mathematics*, vol. 2, no. 1, pp. 35–48, 2017.

[15] M. A. Ramadan and M. R. Ali, "An efficient hybrid method for solving fredholm integral equations using triangular functions," *New Trends in Mathematical Science*, vol. 5, no. 1, pp. 213–224, 2017.

[16] M. R. Ali and A. R. Hadhoud, "Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet scheme for solving the 2D Bratu problem," *Results in Physics*, vol. 12, pp. 525–530, 2019.

[17] R. Sadat and M. Kassem, "Explicit solutions for the (2+ 1)-dimensional jaulent–miodek equation using the integrating factors method in an unbounded domain," *Mathematical & Computational Applications*, vol. 23, no. 1, pp. 1–9, 2018.

[18] R. Gorenflo, "Computation of rough solutions of Abel integral equations," in *Inverse and ill-posed problems*, H. Engl and C. Gioetsch, Eds., pp. 195–210, Academic Press, New York, NY, USA, 1987.

[19] R. Gorenflo and S. Vessella, "Abel Integral Equations: Analysis and Applications," in *Lecture Notes in Mathematics*, vol. 1461, Springer, Berlin, Germany, 1991.

[20] P. K. Lamm and L. Eldén, "Numerical solution of first-kind Volterra equations by sequential Tikhonov regularization," *SIAM Journal on Numerical Analysis*, vol. 34, no. 4, pp. 1432–1450, 1997.

[21] A.-M. Wazwaz, *A First Course in Integral Equations*, World Scientific Publishing Company, New Jersey, NJ, USA, 1997.

[22] B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators*, Institute for Nonlinear Science, Springer, New York, NY, USA, 2003.

[23] T. Kaczorek, *Selected Problems of Fractional Systems Theory*, vol. 411 of *Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin, Germany, 2011.

[24] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies*, vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, 2006.

[25] L. Zhu and Y. Wang, "Numerical solutions of Volterra integral equation with weakly singular kernel using SCW method," *Applied Mathematics and Computation*, vol. 260, pp. 63–70, 2015.


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